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PROGRESS REPORT.

1. The following Presents to the Library have been received :—

   1. *Madras University Calendar for* 1916, Vols. I, II and III.

   2. *Historical Introduction to Mathematical Literature*: by G. A. Miller. New York, Macmillan Co., 1916, pp. xiii+302.

2. On page 24 of Prof. Miller's interesting book, there is a reference to our Society and the Journal in the following terms :—

"In view of the fact that Asia took practically no part in the development of mathematics in modern times up to the beginning of the twentieth century, it is interesting to note another thriving mathematical journal which was started with its borders during the period under consideration. This periodical is entitled *The Journal of the Indian Mathematical Society*, and it was started at Madras, India, in February 1909. The fact that it is the official organ of a society founded in 1907 for the advancement of mathematical study and research in India makes the journal the more interesting and increases its opportunities for usefulness."

Again on page 51 mention is made of the **Indian Mathematical Society** among the **Leading National Mathematical Societies** of the world.

POONA,
31st May 1916.

D. D. KAPADIA,
Hony. Joint Secretary.

The Central Point.

By R. Vythynathaswamy.

[In connection with the present note reference may be made to Bell's *Co-ordinate Geometry* or to Frost's *Solid Geometry* or to Salmon's *Analytical Geometry*, where analytical proofs of the properties of the *central point* will be found. The discussion given here, being mainly geometrical, will be found comparatively simpler].

1. A plane through a generator of a scroll touches the scroll at one and only one point on the generator.

In what follows we shall denote the given generator by PQ, the consecutive generator by P′Q′, where PP′, QQ′ are both ⊥ to PQ. Any plane through PQ cuts P′Q′ in a point which ultimately tends to a definite point in PQ. Hence it is a tangent plane at one and only one point in PQ.

2. The cross-ratio of four points on a generator is equal to the cross-ratio of the tangent planes at these points. (Salmon, § 470).

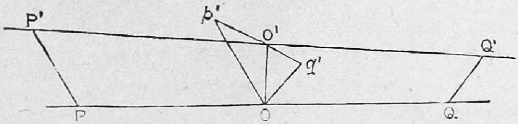
For, by § 1 an (1,1) correspondence exists between tangent planes and the points of contact.

3. The points of contact of perpendicular tangent planes through the same generator are in involution.

For, the pairs of perpendicular tangent planes are themselves in involution.

4. The centre of this involution is *the central point* defined as the point of nearest approach of the given generator and its *consecutive*.

Let OO′ be the shortest distance between PQ and P′Q′. Draw OL parallel to P′Q′. As Q proceeds to ∞ along the line, QQ′ will ultimately lie in the plane POL. Hence QQ′ is ultimately ⊥ to OO′; in other words, since OQQ′ is in the limit the tangent plane at Q, we see that the tangent planes at the central point and the point at ∞ are perpendicular, which proves the theorem.



5. The constant of this involution is equal to the square of the parameter of distribution.

Let P, Q be points on the given generator, the tangent planes at which are ⊥, let P′Q′ be the consecutive generator where PP′, QQ′ are both ⊥ to PQ. Then PP′, QQ′ are mutually perpendicular. Denoting by $p'q'$ the projection of P′ Q′ on a plane through the shortest distance ⊥ to PQ, it is readily seen that the △ Opq', is right-angled at O and that OO′ is the perpendicular on the hypotenuse. Hence, if $OO'=\delta$ and θ be the angle between the generators,

$$\begin{aligned}\delta^2=OO'^2&=p'O'.O'q'\\&=PO.\ OQ.\ \sin^2\theta.\ \sec^2\theta.\\&=PO\cdot OQ\cdot\theta^2.\end{aligned}$$

$$\therefore\qquad OP\cdot OQ=-\frac{\delta^2}{\theta^2}=-p^2.$$

where p is the parameter of distribution.

Cor. If δ_1, δ_2 be the distances between the generators at corresponding points, then from the right-angled △ $Op'q'$

$$\frac{1}{\delta^2}=\frac{1}{\delta_1^2}+\frac{1}{\delta_2^2}.$$

6. The following is another method, substantially the same in principle, of dealing with the central point.

Let x be the distance from a fixed origin of any point in the generator, and $t=\tan\alpha$ be the tangent of the angle made by the normal at this point with a fixed direction ⊥ to the generator. Then since either x or t may be taken as the co-ordinate of a point in the generator, there exists a homographic relation between x and t. Consequently we have the identical relation

$$\begin{vmatrix}1 & x_1 & t_1 & x_1t_1\\ 1 & x_2 & t_2 & x_2t_2\\ 1 & x_3 & t_3 & x_3t_3\\ 1 & x_4 & t_4 & x_4t_4\end{vmatrix}=0.$$

Now choose the fixed direction to be ⊥ to the normal at the point at ∞, the fixed origin to be the point (which we shall call the central point) defined by $t=0$. In the above relation, let x_3, x_4 refer to the origin and the point at ∞ respectively, so that

$$x_3=0,\ t_3=0,\ x_4=\infty,\ t_4=\infty.$$

Substituting, we find

$$\frac{x_1}{t_1}=\frac{x_2}{t_2}=p\ \text{(say)}$$

(1) Putting $t_1 t_2 = -1$, we have $x_1 x_2 = -p^2$, which shews that the points of contact of $\perp$ tangent planes belong to an involution the centre of which is the central point.

(2) This central point is also the point of the nearest approach of the given generator and its consecutive. The relation

$$\frac{1}{\delta^2} = \frac{1}{\delta_1^2} + \frac{1}{\delta_2^2}$$

proved in § 5. *Cor.*, holds for the central point and the point at ∞. Since δ_2 at the point at ∞ is infinite relatively to δ, we must have $\delta_1 = \delta$ or the central point is the point of nearest approach.

(3) The quantity p is the *parameter of distribution.*

For a point very near the central point, it is easily seen by projection, as in § 5, that $t_1 = \frac{x_1 \sin \dot{\theta}}{\delta}$.

$$\therefore \qquad p = \frac{x_1}{t_1} = \frac{\delta}{\dot{\theta}} = \text{parameter of distribution.}$$

7. From the homographic relation between t and x, we readily deduce the following :—

(1) If two scrolls have a common generator and intersect at the same angle α at three points in it, their angle of intersection at any other point is also α.

Let t, t' be the tangents of the angles made by the normals to the scrolls at any point in the generator with a fixed direction $\perp$ to the generator. Since t, t' are each homographically related to x, they must be homographically related to each other. Hence if t' is a certain function of t at *three* points it must be the same function of t at all other points, which proves the theorem.

The theorem includes the particular cases of touching and intersecting orthogonally. It is further evident that the central points and parameters of distribution of the common generator regarded as belonging to the two scrolls are the same.

(2) If two scrolls have a common generator there are two points in the generator at which they cut at a given angle α.

Let $t = \tan \beta$, $t' = \tan \beta'$ represent the tangents of the angles made by the normals to the two scrolls at a common point with a fixed direction $\perp$ to the generator. Let $t'' = \tan (\beta + \alpha)$. Since there is a homographic relation between t and t' and also between t and t'', there holds a relation of the form $t'' = \frac{a\,t' + b}{c\,t' + d}$. Putting $t'' = t'$ in this, we see that there are two points at which the scrolls cut at an angle α.

In particular, there are two points at which they touch and two points at which they cut orthogonally.

(3) *The normals to a scroll at points on a generator generate a hyperbolic paraboloid.* (Salmon, § 472).

Consider the hyperbolic parabolid determined by the normals *at three points* on the generator. This paraboloid has the given generator in common with the scroll and cuts the scroll orthogonally at these three points. Hence it cuts the scroll orthogonally at any other point K in the generator. Thus the tangent plane to the paraboloid at K is the plane containing the generator and the normal at K. But the other generator through K to the paraboloid must lie in the tangent plane and be ⊥ to the given generator. Hence it must be the normal at K. Thus this paraboloid contains the normals at all points in the generator.

(4) *Consecutive normals to the general surface.*

Let R_1, R_2 be the centres of principal curvature at O to a surface, C X,C Y the axes of the indicatrix; C P, C P′ diameters equally inclined to the axes. It is evident from considerations of symmetry that the normals at P, P′ are equally inclined to the normal at O (this is evident if we consider these normals as the normals to the enveloping cone touching the surface along the indicatrix and having its axis along O R_1). Since further C P′ = CP′, it follows that the point of nearest approach (L_1) of normals to the normal at O is the same. Let CQ be the diameter conjugate to CP ; P_1, P_2 the centres of curvature of the normal sections OCP, OCQ. Since the curvatures of the sections OCP, OCP′ are the same, it follows that given L_1 we can find a unique position for P_1 and therefore also for P_2. Hence an (1, 1) correspondence exists between L_1 and P_2. By this correspondence R_1 is carried to R_2 and R_2 to R_1. Since the correspondence carries two points into each other it must be an involution.

Further the shortest distance between the normals at P and O is parallel to C Q for the tangent at P is ⊥ to both the normals. If CT is an asymptote to the indicatrix, it follows (since CT is self-conjugate) that OT is the shortest distance between the normals at O and T. Since the centre of curvature of the section OCT is at ∞, it follows that, in the above involution, the point corresponding to O is at ∞. Thus O is the centre of the involution.

Hence $$OL_1 \cdot OP_2 = OR_1 \cdot OR_2$$

∴ $$OL_1 = \frac{OR_1 \cdot OR_2}{OP_2}.$$

Let α be the eccentric angle of P and let $r_2 = CQ$. Let R be a point in QR_1 such that a line through R parallel to CX meets the normal at P. Since the shortest distance of the normals at P and O is parallel to CQ, it follows that

$$L_1R : L_1O = \tan P\hat{C}X : \tan Q\hat{C}P$$

$$= \frac{b}{a}\cot\alpha : \frac{ab}{a^2 + r_2^2}\cot\alpha$$

$$= \left(1 - \frac{r_2^2}{a^2}\right).$$

$$\therefore \qquad RL_1 = OL_1 - \frac{OP_2}{OR_1} \cdot \frac{OR_1 \cdot OR_2}{OP_2}$$

$$= OL_1 - OR_2.$$

Thus R is same as R_2. *Hence the normal meets the two lines drawn through either centre of curvature perpendicular to the corresponding section.* (Sturm's Theorem).

(5) Defining an asymptotic curve as a curve on the surface, enveloped by inflexional tangents, we have already shewn that the shortest distance between the normals at two consecutive points of an asymptotic curve is the chord joining them. Or the osculating plane of an asymptotic curve is the tangent plane to the surface at the same point.

The torsion of the asymptotic curve at any point is the square root of the specific curvature of the surface at the same point.

The shortest distance between the normal at T and at O is OT and we know by Sturm's theorem, that lines can be drawn through R_1 and R_2 $\perp$ to OR_1 and to each other, to meet the normal at T. Hence if δ θ, be the shortest distance and the angle between the normals, we have

$$\frac{\delta^2}{\theta^2} = OR_1 . OR_2$$

$$\text{Hence, torsion} = \frac{\theta}{\delta} = \text{square root of specific curvature.}$$

(6) We can immediately deduce from this the remarkable property :—

The product of the radii of principal curvature at the central point of a generator of a scroll is equal to the square of the parameter of distribution.

A generator of the scroll is an asymptotic curve and should in this connection be regarded as something like a straight twisted wire,

having a definite osculating plane and binormal at each point. The torsion of this at the point $t = \tan \alpha$ is

$$\frac{dt}{dx}\bigg/\frac{dt}{d\alpha} = \frac{\cos^2\alpha}{p},$$

where p is the parameter of distribution. Hence by the previous theorem the specific curvature at this point is $\frac{\cos^4\alpha}{p^2}$. At the central point $\alpha = 0$, and we have $p^2 = \rho\rho'$

Corollaries :

(i) Consider the scroll generated by the normals along an asymptotic curve. The asymptotic curve will be a line of striction and the parameter of distribution along any normal will be $\frac{1}{\sqrt{S}}$ where S is the corresponding specific curvature. The tangents to this scroll at the centres of principal curvature will be perpendicular.

(ii) The sum of the square roots of the specific curvature at corresponding points of a generator of a scroll is constant and equal to $\frac{1}{p}$.

For, we have $$\frac{\cos^2\alpha}{p} + \frac{\sin^2\alpha}{p} = \frac{1}{p}.$$

(iii) If D is the central ⊥ on a generator of a hyperboloid, axes a, b, c we have

$$D^2 p = abc.$$

For $p^2 = \rho\rho'$ at the central point. Also the tangent plane at the central point is ⊥ to the central plane through the generator. Hence by the well-known properties of the conicoid, if α, β be the axes of the parallel central section

$$p^2 = \rho\rho' = \frac{\alpha^2\beta^2}{D^2} = \frac{a^2b^2c^2}{D^4}.$$

$$\therefore \qquad D^2 p = abc.$$

Japanese Mathematics.

The united labours of Dr. D. E. Smith of New York and Mr. Mikami of Japan, both workers of no mean order in the field of the history of Mathematics, have given to the mathematical world a noteworthy work entitled '*A History of Japanese Mathematics*' (Chicago, 1914). The book is eminently well written and aims at a brief survey of the principal features in the native Japanese mathematics or *wasan*, as it is called. Dr. Smith's scholarly acquaintance with the history of European Mathematics is in evidence throughout the work, a close study of which will amply repay the student.

To us in India, familiar with the traditional methods of dealing with the science of mathematics, the work has a peculiar interest. The Japanese problems are more or less similar to those of *Lilavati* and are inspiring.

The book is printed on excellent paper and is profusely illustrated. It consists of 288 pages divided into fourteen chapters and an index. The reader's interest is well sustained and we make no apology for extracting freely from this unique production. We cannot help regretting, with the authors, the decacy of the *wasan* or native Japanese mathematics, as a result of contact with the West and the modern tendency to broaden the science by bringing it to the level of European mathematics.

The following extracts will speak for themselves :—

(1) "Chin Chiu-shao-(1247) gives a method of approximating the roots of numerical higher equations which he speaks of as the *Ling-lung-kae-fung*, "Harmoniously alternating evolution," a plan in which, by the manipulation of the *sangi*,* he finds the root by what is substantially the method rediscovered by Horner in England in 1819. Another writer of the same period, Yang Hwuy, in his analysis of the *Chiu-chang*, gives the same rule under the name of *Tsang-ching-fang*, "Accumulating involution," but he does not illustrate it by solved problems. We are therefore compelled to admit that Horner's method is a Chinese product of the 13th century, and we shall see that the Japanese adapted it in what we called the third period of their mathematical history.

It is also interesting to know that Chu Chi-chieh in the *Szeyuen Yu-kien* (1303) gives as an "ancient method" the relation of the binomial co-efficients known in Europe as the "Pascal triangle," and that

* **[A calculating device used by the Japanese in which square prisms of 7 mm. thick and 5 cm. long, were used.]**

among his names for the various monads (unknowns) is the equivalent for *thing*. This is the same as the Latin *res* and the Italian *cosa*, both of which had undoubtedly come from the East. It is one of the many interesting problems in the history of mathematics to trace the origin of this term.

Chu Chi-chieh writes the equivalent of $a+b+c+x$ as is here shewn, except that we use T for the symbol *tai*, and the modern numerals instead of the *sangi* forms. The square of this expression he writes thus :

$$\begin{array}{ccc} & 1 & \\ 1 & T & 1 \\ & 1 & \end{array}$$

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & 0 & 2 & \\ 1 & 0 & T^2 & 0 & 1 \\ & 2^2 & 0 & .\,2 & \\ & & 1 & & \end{array}$$

a method that is quickly learned and easily employed.

The single rule laid down in Chu Chi-*chieh's Suan-hsiao Chi-meng* for the use of *sangi* in the solution of numerical equations contains but little positive information. The method is best understood by actually solving a numerical higher equation, but inasmuch as the manipulation of the *sangi* has already been explained in the preceding chapter, the co-efficients will now be represented by modern numerals. The problem which we shall use is taken from the eighth book of the *Tengen Shinan* of Satō Moshun or Shigeharu, published in 1698, and only the general directions will be given, as was the custom. The reader may compare the work with the common Horner method in which the reasoning involved is more clear.

Let it be required to solve the equation

$$11520-432x-236x^2+4x^3+x^4=0.$$

Arrange the *sangi* on the board to indicate the following :

(*r*)					
(0)	1	1	5	2	0
(1)		—	4	3	2
(2)		—	2	3	6
(3)					4
(4)					1

Here the top line, marked (r), is reserved for the root, and the lines marked (0), (1), (2), (3), (4) are filled with the *sangi* representing the co-efficient of 0th, 1st, 2nd, 3rd, 4th powers of the unknown quantity.

First, advance the 1st, 2nd, 3rd and 4th degree classes 1, 2, 3, 4 places respectively, thus :

(r)					
(0)	1	1	5	2	0
(1)	—	4	3	2	
(2)	—2	3	6		
(3)		4			
(4)	1				

The root will have two figures and the tens' figure is 1. Multiply this 10 by the 1 of class (4) and add it to class (3), thus making 14 in class (3). Multiply this 14 by the root 10 and add it to —236 of class (2), thus making —96 in class (2). Multiply this —96 by the root 10, and add to —432 of class (1), thus making —1392 in class (1). Multiply this —1392 by root 10 and add to 11520 of class (0), thus making —2400. The result then appears as follows :—

(r)				1	
(0)		—2	4	0	0
(1)	—1	3	9	2	
(2)		—9	6		
(3)	1	4			
(4)	1				

Now repeat the process, multiplying the root 10 into class (4) and adding to class (3), making 24 ; multiply 24 by the root and add to class

(2), making 144; multiply 144 by the root and add to class (1), making 48. The result then appears as follows:

(r)				1	
(0)		−2	4	0	0
(1)			4	8	
(2)	1	4	4		
(3)	2	4			
(4)	1				

Repeat the process, multiplying the root 10 into class (4) and adding to class (3), making 34; multiply 34 by the root and add to class (2) making 484.

Again repeat the process, multiplying the root into class (4) and adding to class (3), making 44.

Now move the *sangi* representing the co-efficients of classes (1). (2) (3), (4) to the right 1, 2, 3, 4 places, respectively, and we have:

(r)				1	
(0)		−2	4	0	0
(1)				4	8
(2)			4	8	4
(3)				4	4
(4)					1

The second figure of the root is 2*. Multiply this into class (4) and add to class (3), making 46. Multiply the same root figure, 2, into this class (3) and add to class (2), making 576. Multiply this 576 by

* It is not stated how either figure is ascertained.

2 and add to class (1), making 1200. Multiply this 1200 by 2 and add to class (0), making 0. The work now appears as follows:

(r)				1	2
(0)					
(1)		1	2		
(2)			5	7	6
(3)				4	6
(4)					1

The root therefore is 12.

It may now be helpful to give a synoptic arrangement of the entire process in order that this Chinese method of the 13th century, practised in Japan in the 17th century, may be compared with Horner's method. The work as described is substantially as follows:

Given

$$x^4+4x^3-236x^2-432x+11520=0.$$

$$\begin{array}{l}
1+4-236-432+11520 \\
\quad\ 10\quad 140-960-13920 \\
\hline
1\ 14-\ 96-1392-2400 \\
\quad\ 10\quad 240\quad 1440 \\
\hline
1\ 24\ 144\quad 48-2400 \\
\quad\ 10\ 340\qquad\quad 2400 \\
\hline
1\ 34\ 484\quad 48\qquad 0 \\
\quad\ 10\qquad 1152 \\
\hline
1\ 44\ 484\quad 1200 \\
\quad\quad 2\quad 92 \\
\hline
1\ 46\ 576
\end{array}$$

Chu Chi-chieh also gives rules for the treatment of negative numbers." [pp. 50—56]

(2) "The name of Seki (1642—1708) has long been associated with the *Yenri*, a form of the calculus that was possibly invented by him. It is with greater certainty that he is known for his *tenzan* method, an algebraic system that improved upon the method of the 'celestial element' inherited from the Chinese; ~~for the *Yendan jutsu*, a scheme~~ by which the treatment of equations and other branches of algebra is

simpler than by the methods inherited from China ; and for his work in determinants that ante-dated what has hitherto been considered the first discovery, namely the investigations of Leibnitz." [pp. 94—95].

(3) "Mention should be made of Seki's work on the mensuration of solids, which appeared in two of his manuscripts. He begins by considering the volume of a ring (*kokan* in Japanese), generated by the rovolution of a segment of a circle about a diameter parallel to the chord of the segment. He states that the volume is equal to the product of the cube of the chord and the moment of spherical volume (that is, the volume of a unit sphere).

He likewise finds the volume generated by a lune formed by two arcs, the axis being parallel to the common chord. Such work does not seem very difficult at present, but in Seki's time it was an advance over anything known in Japan. These problems were to Japan what those of Cavalieri were to Europe, making a way for the *Katsujutsu* or the method of multiple integration of a later period. * * *

One of the most marked proofs of Seki's genius is seen in his anticipation of the notion of determinants. The school of Seki offered in succession five diplomas representing various degrees of efficiency. The diploma of the third class was called the *Fukudai-menkyo* and represented eighteen or nineteen subjects. The last of these subjects related to the *fukudai* problems or problems involving determinants, and since it appears in a revision of 1683, its discovery antedates this year. Leibnitz (1646—1716) to whom the Western world generally assigns the first idea of determinants, simply asserted that in order that the equations

$$10+11x+12y=0,\ 20+21x+22y=0,\ 30+31x+32y=0$$

may have the same roots the expression

$$10.21.32-10.22.31-11.20.32+11.22.30+12.20.31-12.21.30$$

must vanish. On the other hand Seki treats of n equations. While Leibnitz's discovery was made in 1693 and was not published until after his death, it is evident that Seki was not only the discoverer but that he had a much broader idea than that of his great German contemporary. To show the essential features of his method we may first suppose that we have two equations of the second degree,

$$ax^2+bx+c=0$$
$$a'x^2+b'x+c'=0.$$

Eliminating x^2 we have

$$(a'b-ab')x+(a'c-ac')=0,$$

and eliminating the absolute term and suppressing the factor x we have

$$(ac'-a'c)x+(bc'-b'c)=0.$$

That is, we have two equations of the second degree and transform them into two equations of the first degree, by what the Japanese called the process of folding (*tatamu*). In the same way we may transform n equations of the nth degree into n equations of the $(n-1)$ the degree. From the latter equations the *wasanka* (the follower of the *wasan* or native mathematics) proceeded to eliminate the various powers of x. Since it was their custom to write only the co-efficients, including all zero co-efficients, and not to equate to zero, their array of co-efficients formed in itself a determinant, although they did not look upon it as a special function of the co-efficients. On this array Seki now proceeds to perform two operations, the *san* (to cut) and the *chi* (to manage). The *san* consisted in the removal of a constant literal factor in any row or column, exactly as we remove a factor from a determinant to-day. If the array(our determinant)equalled zero, this factor was at once dropped. The *chi* was the same operation with respect to a numerical factor.

Seki also expands this array of co-efficients, practically the determinant that is the eliminant of the equations. In this expansion some of the products are positive and these are called *sei* (kept alive), while others are negative and are called *koku* (put to death), and rules for determining these signs are given. Seki knew that the number of terms in the expansion of a determinant of the nth order was $n!$, and he also knew the law of interchange of columns and rows. Whatever, therefore, may be our opinion as to Seki's originality in the *yenri*, or even as to his knowledge of that system at all, or as to its value, we are compelled to recognize that to him rather than to Leibnitz is due the first step in the theory which afterwards, chiefly under the influence of Cramer (1750) and Cauchy (1812) was developed into the theory of determinants." [pp. 121—6.]

(4) "Takebe (1664—1739) states that Seki was wont to say that calculations relating to the circle were so difficult that there could be no general method of attack. Indeed he says that Seki was averse to complicated theories, while he himself, took such delight in minute analysis that he finally succeeded in his efforts at the quadrature of the circle. It would thus appear that the *yenri* was not the product of Seki's thought, but rather of Takebe's painstaking labour. * * *

He has

$$\frac{1}{4}a^2=dh\left[1+\sum_{1}^{\infty}\frac{2^{2n+1}(n!)^2}{(2n\times 2)!}\left(\frac{h}{d}\right)^n\right],$$

for the square of half an arc a in terms of the height h and diameter d of the arc." [p. 147.]

(5) "Matsunaga's *Hōyen Sankyō* (1739) is composed of five books and is devoted entirely to formulas for the circumference and arcs of a circle, no analysis appearing. His first series is as follows :—

$$\frac{\pi^2}{9}=1+\frac{1^2}{3.4}+\frac{1^2.2^2}{3.4.5.6}+\frac{1^2.2^2.3^2}{3.4.5.6.7.8}+\cdots$$

This is followed by

$$\frac{\pi}{3}=1+\frac{1^2}{4.6}+\frac{1^2.3^2}{4.6.8.10}+\frac{1^2.3^2.5^2}{4.6.8.10.12.14}+\cdots$$

a series which is then employed for the evaluation of π to fifty figures. The result is the following :

$$\pi=3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 41971\ 69399\ 5751.$$

[p. 160.]

(6) " Ajima (1739—1798) made a noteworthy change in the *yenri*, in that he took equal divisions of the chord instead of the arc, thus simplifying the work materially. Indeed we may say that in this work Ajima shows the first real approach to a mastery of the idea of the integral calculus that is found in Japan, which approach we may put at about the year 1775. * * *

He gets the length of the arc as follows :

$$\text{arc}=c+\frac{1^2}{2.3}\frac{c^3}{d^2}+\frac{1^2.3^2}{2.3.4.5}\frac{c^5}{d^4}+\frac{1^2.3^2.5^2}{2.3.4.5.6.7}\frac{c^7}{d^6}+\cdots$$

from which other formulas may be derived.

* * * *

Thus we at last find in Ajima's work, the calculus established in the native Japanese mathematics, although possibly with considerable European influence. With him the use of the double series again appears, it having been already employed by Matsunaga and Kurushima, and by him the significance of double integration seems first to have been realized. He lacked the simple symbolism of the West, but he had the spirit of the theory, and although his contemporaries failed to realize his genius in this respect it is now possible to look back upon his work and to evaluate it properly. As a result it is safe to say that Ajima brought mathematics to a higher plane than any other man in the eighteenth century, and that had he lived where he could easily have come into touch with contemporary mathematical thought in other parts of the world he might have made discoveries that would have been of far-reaching importance in the science." [pp. 201—205]

(7) "Wada Nei (1787—1840) also turned his attention to the computation of volumes, simplifying Ajima's work on the two intersecting

cylinders and in general developing a very good working type of the integral calculus so far as it has to do with the question of mensuration.

The question of maxima and minima had already been considered by Seki more than a century before Wada's time, the rule employed being not unlike the present one of equating a differential coefficient to zero, although no explanation was given for the method. * * * He (Wada) not only gave the reason for the rule, but carried the discussion still further, including in his theory, the subject of the maximum and minimum values of infinite series." [pp. 224—5]

(8) "Thus closes the old *wasan*, the native mathematics of Japan. It seems as if a subconscious feeling of the hopelessness of the contest with Western science must have influenced the last half century preceding the opening of Japan. There was no really worthy successor of Wada Nei in all this period, and the feeling that was permeating the political life of Japan, that the day of isolations was passing, seems also to have permeated scientific circles." [p. 253]

(9) "At the close of the eighteenth century Shizuki Tadao (1760—1806) began a work entitled *Rekisho Shinsho* consisting of three parts, each containing two books, the composition of which was completed in 1803. The treatise which was never printed is based upon the works of John Keill (1671—1721) Professor of Astronomy at Oxford. The first part treated of the Copernican system of astronomy and the second and third parts of mechanical theories. The latter part of the work may have had its inspiration from Newton's *Principia*. It was the first Japanese work to treat of mechanics and physics, and it is noteworthy also from the fact that the appendix to the third part contains a nebular hypothesis that is claimed to have been independent of that of Kant and Laplace." [p. 263]

(10) "Europe had several thousand years of mathematics back of her when Newton and Leibnitz worked on the calculus,—years in which every nation knew or might know what its neighbours were doing; years in which the scholars of one country inspired those of another. Japan had had hardly a century of real opportunity in mathematics when Seki entered the field. From the standard of opportunity Japan did remarkable work; from the standpoint of mathematical discovery this work was in every way inferior to that of the West.

When, however, we come to execution it is like picking up a box of the real old red lacquer,—not the kind made for sale in our day. In

execution the work was exquisite in a way wholly unknown in the West. For patience, for the everlasting taking of pains, for ingenuity in untangling minute knots and thousands of them, the problem-solving of the Japanese and the working out of some of the series in the *yenri* have never been equalled.

And what will be the result of the introduction of the new (occidental) mathematics into Japan? It is altogether too early to foresee, just as we cannot foresee the introduction of new art, of new standards of living, of machinery, and of all that goes to make the New Japan. If it shall lead to the application of the peculiar genius of the old school, the genius for taking infinite pains, to large questions in mathematics, then the world may see results that will be epoch-making. If, on the other hand, it shall lead to a contempt for the past, and to a desire to abandon the very thing that makes the *wasan* worthy of study, then we cannot see what the future may have in store." [p. 280.]

SHORT NOTES.

The Period of the Elliptic Functions sn u, cn u, dn u.

1. The period K of the above elliptic functions is known to be equivalent to the series

$$\frac{\pi}{2}\left\{1+\left(\tfrac{1}{2}\right)^2 k^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2 k^4+\ldots\right\} \quad \ldots \quad \ldots \quad (1)$$

where k is the modulus. [Dixon. *Elliptic Functions*, p. 79.]

This series is the same as $\frac{1}{2}\pi\ F(\frac{1}{2}, \frac{1}{2}, 1, k^2)$, with the usual notation F for a hypergeometric series.

Also, from Gauss' property of the function F, we have

$$(1+y)^{2\alpha} F\{\alpha, \alpha+\tfrac{1}{2}-\beta, \beta+\tfrac{1}{2}, y^2\}=F\left\{\alpha, \beta, 2\beta, \frac{4y}{(1+y)^2}\right\},$$

[Forsyth : *Diffl. Equations*, p. 214, Ex. 5 (ii)]

so that, putting $\alpha=\beta=\frac{1}{2}$, $y=k$, we infer that

$$(1+k)\ F(\tfrac{1}{2}, \tfrac{1}{2}, 1, k^2)=F\left\{\tfrac{1}{2}, \tfrac{1}{2}, 1, \frac{4k}{(1+k)^2}\right\}$$

Hence the period K for modulus k is connected with the period P for modulus $p=\dfrac{2\sqrt{k}}{1+k}$ by the following simple relation

$$(1+k)K=P. \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

In other words, the series for the period admits of transformation into

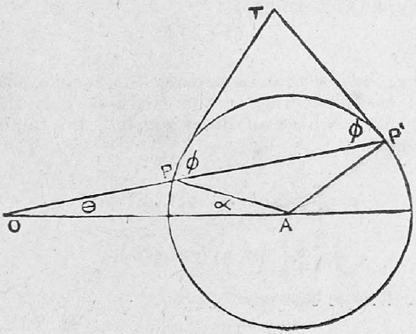
$$\frac{\pi}{4}(1+\sqrt{1-p^2})\left\{1+(\tfrac{1}{2})^2p^2+\left(\frac{1\cdot3}{2\cdot4}\right)^2p^4+\ldots\right\} \quad \ldots \quad \ldots \quad (3)$$

2. The following Dynamical proof of the above property is worthy of notice :

Consider the gravitation potential of a thin uniform circular ring (centre A, radius a) at a point O, distant $c>a$ from the centre. With the usual notation, we have

$$V=\mu\nu\left[\int\frac{ds}{r}+\int\left(\frac{ds'}{r'}\right)\right]$$

where ds, ds' are elements at P, P′.



Now

$$\sin\phi=\frac{rd\theta}{ds},\ \sin\phi'=\frac{r'd\theta}{ds'};$$

$\therefore$
$$\frac{ds}{r}=\frac{ds'}{r'}$$

$\therefore$
$$V=2\mu\nu\int\frac{ds}{r}=2\mu\nu\int\frac{d\theta}{\sin\phi}$$

$$=-2\mu\nu\int\frac{ad\phi}{b\cos\theta},\ \text{since } a\cos\phi=c\sin\theta,$$

$$=-2\mu\nu\int\frac{kd\phi}{(1-k^2\cos^2\phi)^{\frac{1}{2}}},$$

where $k=a/c$ and the integration extends over the arc of the ring intercepted between the tangents from O to it.

Hence
$$V=4\mu\nu\, k.K, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$
integrating between proper limits.

Again, from the formula $r^2=a^2+c^2-2ac\cos\alpha$,

$$V=\mu\nu\int_C\frac{ds}{r}=2\mu\nu\int_0^{\pi}\frac{ad\alpha}{(a^2+c^2-2ac\cos\alpha)^{\frac{1}{2}}}$$

$$=4\mu\nu\int_0^{\frac{1}{2}\pi}\frac{ad\lambda}{[(a+c)^2-4ac\cos\lambda^2]^{\frac{1}{2}}},\ [2\lambda=\alpha]$$

$$=4\mu\nu\int_0^{\frac{1}{2}\pi}\frac{ad\lambda}{(a+c)(1-p^2\cos^2\lambda)^{\frac{1}{2}}}$$

$$=4\mu\nu\, k\, P/(1+k), \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

writing $p=\dfrac{2\sqrt{ac}}{a+c}=\dfrac{2\sqrt{k}}{1+k}$.

Equating (4) and (5), we get

$$P=(1+k)\, K$$

which is result (2).

(3) Further, we see that, according to Jacobi's construction, the limiting point L_1 corresponding to the modulus p is the focus of the ellipse of eccentricity k whose auxiliary circle is the fundamental circle of the construction. [Dixon, § 105.]

For

$$p=\frac{2\sqrt{e_1}}{1+e_1},\ \text{if } e_1=\Omega L_1,\ \Omega A$$

$$=\frac{2\sqrt{k}}{1+k}\ \text{by hypothesis.}$$

Hence $e_1=k$, whence the result stated.

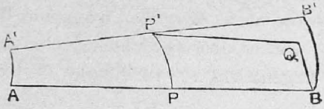
M. T. NARANIENGAR.

Holditch's Theorem.

1. Various proofs of this elegant theorem are found in Text-books on *Integral Calculus*. [Williamson, § 147 ; Gibson, p. 323 ; Carr, 5244.]

The following demonstration may interest the reader on account of its simplicity :

Let a line P B of length b move to a new position P′B′, where P P′, BB′ are elements of the paths of P and B. Then, resolving the motion of PB into a translation and a rotation, we have



$$\text{area } \mathrm{P\,B\,B'P'} = \square\,\mathrm{PQBP'} + \triangle \mathrm{P'QB'}$$
$$= ds.b \sin\phi + \tfrac{1}{2}\, b^2\, d\,\theta, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where ds denotes the element PP′, ϕ the angle A PP′ and $d\theta$ the rotation of BP.

Integrating (1), the area swept over by P B is

$$\mathrm{S} = b\int \sin\phi\, ds + \tfrac{1}{2}\int b^2 d\theta. \quad \ldots \quad \ldots \quad (2)$$

If P and B describe complete closed curves of areas (P), (B), one inside the other, obviously

$$\mathrm{S} = (\mathrm{B}) - (\mathrm{P}) \quad \ldots \quad \ldots \quad (3)$$

Hence from (2) and (3)

$$(\mathrm{B}) - (\mathrm{P}) = b\int_{\mathrm{P}} (ds \sin\phi) + \frac{1}{2}\, b^2 \int_0^{2\pi} d\theta$$
$$= b \cdot \mathrm{I} + \pi b^2, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

where I is a quantity depending on the curve (P) and the angle ϕ.

Now, if we take PA $= a$ on BP produced, reasoning as before, we should get

$$(\mathrm{P}) - (\mathrm{A}) = a.\mathrm{I} - \pi a^2. \quad \ldots \quad (5)$$

Eliminating I between (4) and (5), we have

$$a(\mathrm{B}) + b(\mathrm{A}) - (a+b)(\mathrm{P}) = \pi ab(a+b), \quad \ldots \quad (6)$$

which is Holditch's Theorem.

2. In the above demonstration suppose PB, AP are variable and that PB : AP $= b : a$ always. Then P may be regarded as the centre of

masses a, b placed at B, A respectively. Denoting PB, AP by r_2, r_1, we have the formulae

$$(B)-(P)=\int_P r_2 \sin\phi . ds+\tfrac{1}{2}\int_0^{2\pi} r_2^2 d\Theta,$$

$$(P)-(A)=\int_P r_1 \sin\phi . ds-\tfrac{1}{2}\int_0^{2\pi} r_1^2 d\Theta$$

Since $r_1 : r_2 = a : b$, we deduce

$$a(B)+b(A)-(a+b)(P)=aR_2+bR_1 \quad \ldots \quad \ldots \quad (7)$$

where R_1, R_2 denote the areas described by A and B relative to P.

3. From (7) we get at once the 'Theorem on Areas' mentioned by Mr. R. Vythynathaswamy in a previous issue of this Journal (Vol. VII, p. 96), *viz.*

"If a number of particles $a_1, a_2 \ldots a_n$ of masses $m_1, m_2 \ldots m_n$ describe closed curves of areas $A_1, A_2, \ldots A_n$, then the area G described by their centre of mass is given by

$$G.\Sigma(m_1)=\Sigma(m_1 A_1)-\Sigma(m_1 R_1),$$

where $R_1, R_2 \ldots$ are the areas described by $a_1, a_2 \ldots$ relative to the centre of mass."

4. In this connection the following general property may be noticed:

If a vector AB $(=\rho)$ move according to any law with its ends on given curves $r_1=f_1(\Theta_1)$, $r_2=f_2(\Theta_2)$; then the element of area described by the vector is approximately

$$\begin{aligned}\delta S &= \text{area } ABB'A' = OAB+OBB'+OB'A'+OA'A \\ &= \tfrac{1}{2}\{(\rho+\delta\rho)(p+\delta p)-\rho p\}+\tfrac{1}{2}r_2^2\delta\Theta_2-\tfrac{1}{2}r_1^2 d\Theta_1, \\ &= \tfrac{1}{2}(p\delta\rho+\rho\delta p)+\tfrac{1}{2}r_2^2\delta\Theta_2-\tfrac{1}{2}r_1^2\delta\Theta_1,\end{aligned}$$

where p denotes the perpendicular from the origin on the vector.

Integrating

$$\begin{aligned}S &= \tfrac{1}{2}\int(p\,d\rho+\rho dp)+\tfrac{1}{2}\int r_2^2 d\Theta_2-\tfrac{1}{2}\int r_1^2 d\Theta_1 \\ &= (B)-(A),\end{aligned}$$

in the case of closed curves.

Hence, in this case,

$$\int(pd\rho+\rho dp)=0.$$

M. T. NARANIENGAR.

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Note on Question 567.

[The Journal for October 1915 reproduces the clumsy solution of this question found in the text-books. The following discussion of the problem may be of interest to readers of the Journal.]

Let PQ, RS be opposite edges of a uniform tetrahedron of mass M, and let their mid-points be U, V and their lengths $2a$, $2b$; let the length of UV be $2c$, and let G be the mid-point of UV. A plane parallel to PQ and RS, cutting UV in a point whose distance from G towards U is ct, cuts the surface of the tetrahedron in a parallelogram of sides $a(1+t)$, $b(1-t)$, with angles independent of t. Since a prallelogram of mass m is equimomental with particles of masses $m/12$ at the vertices and a particle of mass $2m/3$ at the centre, the tetrahedron is equimomental with a distribution of varying line-density, along the five lines PR, PS, QR, QS, UV, the density in each line being proportional to $1-t^2$, and the total mass of each of the first four lines being $M/12$ and of the fifth line being $2M/3$. Since

$$\int_{-1}^{1} k(1-t^2)dt=4k/3, \int_{-1}^{1} k(1-t^2)t^2\,dt=4k/15,$$

the part of a line of density $k(1-t^2)$ which corresponds to values of t between -1 and 1 has mass n if k is equal to $3n/4$, and the line is equimomental with three particles, one of mass $n/10$ at each end and one of mass $4n/5$ at the mid-point. It follows at once that

'*A uniform tetrahedron of mass M is equimomental with a system of eleven particles, one of mass $M/60$ at each vertex, one of mass $M/15$ at the mid-point of each edge, and one of mass $8M/15$ at the centroid.*'

The deduction of the familiar systems with five particles, of which four are at the vertices, and with seven particles, of which six are at the mid-points of the edges, requires only applications of the theorem that a sytem of three equal particles of mass $m/3$ at the mid-points of the sides of a triangle is equimomental with a system of four particles, one of mass $m/12$ at each vertex and one of mass $3m/4$ at the centroid.

It is evident that the method used here is applicable to many other problems, and it is interesting to use it in the case of a triangle. A uniform line of mass m is equimomental with particles of mass $m/6$ at its end-points and a particle of mass $2m/3$ at its mid-point, and the integrations of t, t^2, and t^3 from 0 to 1 are sufficient to show that a line PQ of mass n whose density is proportional to distance from P has its centroid at the point of trisection nearer to Q and is equimomental with three particles, one of mass $n/12$ at P, one of mass $n/6$ at Q, and one of mass $3n/4$

at the centroid. It follows that a triangle ABC of mass M is equimomental with a system of seven particles, one of mass $M/12$ at A, two of mass $M/36$ at B and C, one of mass $M/9$ at the mid-point of BC, two of mass $M/8$ at the points of trisection of AB, AC which are further from A, and one of mass $M/2$ at the centroid of the triangle. Superposing three distributions of this form each with total mass $M/3$, we find a symmetrical system composed of thirteen particles, one of mass $5M/108$ at each vertex, one of mass $M/27$ at the mid-point of each side, one of mass $M/24$ at each point of trisection of each side, and one of mass $M/2$ at the centroid, and this system can be replaced immediately by a system of seven particles, one of mass $M/18$ at each vertex, one of mass $M/9$ at the mid-point of each side, and one of mass $M/2$ at the mid-point.

ERIC H. NEVILLE.

The Face of the Sky for July and August 1916.

The Sun

enters Leo on July 22 and Virgo on August 23.

Phases of the Moon.

		July.			*August.*		
		D.	H.	M.	D.	H.	M.
First Quarter	...	8	5	25 P. M.	7	3	35 A. M.
Full Moon	...	15	10	10 A. M.	13	5	30 P. M.
Last Quarter	...	22	5	3 A. M.	20	6	23 P. M.
New Moon	...	30	7	45 A. M.	28	10	54 P. M.

Eclipses.

There is a lunar eclipse on July 15 and an annular solar eclipse on July 30—both invisible in India.

The Planets.

Mercury which is a morning star in July is in superior conjunction on July 28 and is an evening star in August. It is in conjunction with Venus on July 14, with Saturn on July 22, with the Moon on July 30 and on August 31, with η Gemini on July 12, with η Caneri on July, 28 and with Neptune on July 27.

Venus is in inferior conjunction on July 4 and is stationary on July 25. It attains its greatest brilliance on August 26. It is in conjunction with the Moon on July 27 and on August 24.

Mars is in conjunction with the Moon on July 6 and on August 4.

Jupiter is in quadrature on July 27. It is stationary on August 25. It is in conjunction with the Moon on July 22 and on August 19 at about 5-30 A. M.

Saturn is in conjunction on July 13. It is in conjunction with the Moon on July 29 and on August 25.

Uranus is in opposition on August 10. It is in conjunction with the Moon on July 17 and on August 13.

Neptune is in conjunction on July 25. It in conjunction with the Moon on July 2, on July 29 and on August 26.

V. RAMESAM.

SOLUTIONS.

Question 245.

(H. Br.) Two curves S, S′ are such that if P is any point on S and P′ on S′,

$$P P' = f_1(t) + f_2(t')$$

where t, t' are the variables in terms of which P, P′ may be expressed respectively. Shew that the curves are the focal conics of a system of confocal quadrics.

Solution by N. Durai Rajan.

Let (X, Y, Z) be the co-ordinates of a point P on the 1st curve, expressed in terms of a single variable T. Similarly let x, y, z, t refer to the 2nd curve. Let PQ, pq be the tangents at P, p and let

$\theta = \angle$ between PQ, Pp
$\phi = \angle$.........pq, pP
$\psi = \angle$.........pq, PQ
(PQ, pq are not coplanar)

By hypothesis

$$(X-x)^2+(Y-y)^2+(Z-z)^2 \equiv R^2 \equiv [F(T)+f(t)]^2 \qquad \dots \quad (1)$$

Differentiating

$$\therefore \quad \Sigma(X-x)\frac{dX}{dT} = R\frac{\partial R}{\partial T} = R.\ F'(T) \qquad \dots \qquad \dots \quad (2)$$

$$\therefore \quad \sum \frac{X-x}{R}.\frac{dX}{dT}.\frac{1}{\sqrt{\left(\frac{dX}{dT}\right)^2+\dots+\dots}} = \frac{F'(T)}{\sqrt{X'^2+Y'^2+Z'^2}}.$$

Hence cos θ is a function of T, and θ is constant if T is constant.

Thus if P be kept fixed and p varies, then the cone described by Pp is right circular, its axis being PQ.

Hence both the curves are curves lying on right circular cones.

Differentiating (2) with regard to T, we get

$$\Sigma(X-x)X''+\Sigma X'^2 = [F'(T)]^2 + F''(T)[F(T)+f(t)]$$
$$= [F'(T)]^2 + \frac{F''(T)}{F'(T)}\Sigma(X-x)X'$$

$$\therefore \quad \Sigma(x-X)\left(X''-X'.\frac{F''(T)}{F'(T)}\right) = \Sigma X'^2 - [F'(T)]^2; \qquad \dots \quad (3)$$

i.e. if P be kept fixed, p lies always in a plane.

Hence the curves are plane curves and lie on quadric cones.

In other words, the curves are conics.

We know that the locus of the vertices of right circular cones standing on a given conic is another conic in a perpendicular plane and these conics are the focal conics of a confocal system of quadrics ; whence, we infer that the given curves are the focal conics of a set of confocal quadrics.

Verification : The conics are

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1,\ z=0: \quad \frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1,\ y=0\ ;$$

$$(X, Y, Z)=\sqrt{a^2-c^2}\cos t,\ \sqrt{b^2-c^2}\sin t,\ 0,$$

$$(x, y, z)=\sqrt{a^2-b^2}\cosh t',\ 0,\ \sqrt{b^2-c^2}\sinh t''.$$

$$\therefore\ R^2=(\sqrt{a^2-c^2}.\cos t-\sqrt{a^2-b^2}\cosh t')^2+(b^2-c^2)\sin^2 t$$
$$+(b^2-c^2)\sinh^2 t'$$

$$=\cos^2 t\,(a^2-c^2-b^2+c^2)+\cosh^2 t'\,(a^2-b^2+b^2-c^2)$$
$$-2\cosh t'\cos t\sqrt{a^2-b^2},\ \sqrt{a^2-c^2}+(b^2-c^2)-(b^2-c^2)$$

$$=\left[\sqrt{a^2-b^2}.\cos t-\sqrt{a^2-c^2}.\cosh t'\right]^2.$$

By equating $\dfrac{\partial^2 R}{\partial T\partial t}=0$, we get on reduction the theorem

$$\cos\psi=\cos\theta.\cos\phi.$$

Question 353.

(S. Ramanujan) :—If n is any positive odd integer shew that

$$\int_0^\infty \frac{\sin nx}{\cosh x+\cos x}\,\frac{dx}{x}=\frac{\pi}{4}$$

and hence prove that

$$\frac{\pi}{8}=\frac{1}{\left(\cosh\frac{\pi}{2n}+\cos\frac{\pi}{2n}\right)}-\frac{1}{\left(\cosh\frac{3\pi}{2n}+\cos\frac{3\pi}{2n}\right)}+\ldots$$

for all odd values of n.

Solution by A. C. L. Wilkinson, M. A., F. R. A. S.

To evaluate

$$\int \frac{\sin x}{\cosh x+\cos x}\,\frac{dx}{x}.$$

Consider

$$\int \frac{e^z - e^{iz}}{e^z + e^{iz}} \frac{dz}{z}$$

taken over the following contour; (1) the x-axis from O to R, (2) the quadrant of the circle centre the origin and radius R, (3) the y-axis from R to O, where R tends to infinity. The infinities of the function are given by

$$z(1-i) = (2n+1)\, i\pi, \text{ or } 2z = -(2n+1)\,\pi\,(1-i);$$

and all these lie on the line $y + x = 0$.

Let (α, β) be any point on the circular quadrant, so that

$$\alpha + i\beta = \mathrm{R}e^{i\theta},$$

then

$$\int_0^\infty \frac{e^x - e^{ix}}{e^x + e^{ix}} \frac{dx}{x} + \int_0^{\frac{1}{2}\pi} \frac{e^{\alpha+i\beta} - e^{-\beta+i\alpha}}{e^{\alpha+i\beta} + e^{-\beta+i\alpha}}\, i d\theta - \int_0^\infty \frac{e^{ix} - e^{-x}}{e^{ix} + e^{-x}} \frac{dx}{x} = 0.$$

The function in the second integral is equal to 1 everywhere on the boundary of the infinite quadrant, whence, collecting the first and third integrals together, we have

$$\int_0^\infty \frac{\sin x}{\cosh x + \cos x} \frac{dx}{x} = \frac{\pi}{4}.$$

For the second part it is sufficient to establish that

$$\int_0^\infty \frac{2 \cos 2rx \sin x}{\cosh x + \cos x} \frac{dx}{x} = 0$$

for integral values of r.

Consider

$$\int_0^\infty \frac{\sin x\,(1 - \cos 2rx)}{\cosh x + \cos x} \frac{dx}{x}.$$

The integral is the same as

$$\int_0^\infty \frac{2\,(1 - \cos 2rx)}{x} \{ e^{-x} \sin x - e^{-2x} \sin 2x + e^{-3x} \sin 3x - \dots \}\, dx$$

$$= \sum_{n=1}^{n=\infty} \int_0^\infty (-)^{n-1} \cdot \frac{e^{-nx}}{x} \cdot \{ 2 \sin nx - \sin (2r+n)x + \sin (2r-n)\,x \}\, dx$$

$$= \sum_{n=1}^{n=\infty} (-)^{n-1} \left\{ \frac{\pi}{2} - \tan^{-1}\frac{2r+n}{n} + \tan^{-1}\frac{2r-n}{n} \right\}$$

$$\text{since } \int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx = \tan^{-1}\frac{b}{a}$$

$$= \sum_{n=1}^{n=\infty} (-)^{n-1} \left\{ \frac{\pi}{2} - \tan^{-1}\frac{n^2}{2r^2} \right\}$$

$$= \sum_{n=1}^{n=\infty} (-)^{n-1} \tan^{-1}\frac{2r^2}{n^2} = \frac{\pi}{4} \text{(Bromwich, \textit{Infinite Series}, p. 259, Ex. 34)},$$

whence

$$\int_0^\infty \frac{\sin x \cos 2rx}{\cosh x + \cos x}\frac{dx}{x} = 0, \text{ and } \int_0^\infty \frac{\sin nx}{\cosh x + \cos x}\frac{dx}{x} = \frac{\pi}{4}, \text{ for } n \text{ odd.}$$

Consider

$$\int \frac{1}{\left(\cosh\frac{z}{n} + \cos\frac{z}{n}\right)}\left(\frac{1}{\cosh z} + \frac{1}{\cos z}\right)\frac{dz}{z}$$

where n is an odd integer. This remains finite for the points at which $\cosh\frac{z}{n} + \cos\frac{z}{n}$ vanishes, since when n is odd $\left(\cosh\frac{z}{n} + \cos\frac{z}{n}\right)$ is a factor of $\cosh z + \cos z$.

Thus the infinities of the function we are integrating are

$$z = 0,\ z = \pm\frac{(2r+1)\pi}{2},\ z = \pm\frac{(2r+)i\pi}{2}$$

where r is a positive integer.

Integrate over the following contour

(1) an infinitestinal circle surrounding $z=0$

(2) circles surrounding $z = \pm\frac{(2r+1)\pi}{2}$ and $z = \pm\frac{(2r+1)i\pi}{2}$

(3) a circle of infinitely great radius R so chosen as not to pass indefinitely near the points $\frac{(2r+1)\pi}{2}$, $\frac{(2r+1)i\pi}{2}$, $\frac{(2s+1)\pi}{2}(1+i)$.

The last condition is required in the discussion of the integral over the circle of radius R even though it is not a point at which the inte-

gral becomes infinite. That this is possible is easily seen. For take $R=k\pi$, where k is an integer and suppose $k\pi=\frac{(2s+1)\pi}{2}\sqrt{2}+\epsilon$, then $R=(k+1)\pi$ satisfies our conditions.

The integral round the circle $z=\frac{(2r+1)\pi}{2}$ gives

$$2\pi i\left\{\frac{1}{\cosh\frac{(2r+1)\pi}{2n}+\cosh\frac{(2r+1)\pi}{2n}}\cdot\frac{-1}{\sin\frac{(2r+1)\pi}{2}}\cdot\frac{1}{\frac{(2r+1)\pi}{2}}\right\}$$

$$\frac{4i}{2r+1}\cdot\frac{1}{\cosh\frac{(2r+1)\pi}{2n}+\cos\frac{(2r+1)\pi}{2n}}.$$

The integral round the circle $z=\frac{(2r+1)i\pi}{2}$ gives

$$2\pi i\left\{\frac{1}{\cos\frac{(2r+1)\pi}{2n}+\cosh\frac{(2r+1)\pi}{2n}}\cdot\frac{1}{\sinh\frac{(2r+1)\pi i}{2}}\cdot\frac{1}{\frac{(2r+1)\pi i}{2}}\right\}$$

$$=(-)^{r-1}\cdot\frac{4i}{2r+1}\cdot\frac{1}{\cosh\frac{(2r+1)\pi}{2n}+\cos\frac{(2r+1)\pi}{2n}}.$$

The integral round $z=0$ gives $2\pi i$.

Denoting by S the series

$$\frac{1}{\cosh\frac{\pi}{2n}+\cos\frac{\pi}{2n}}-\frac{1}{3}\frac{1}{\cosh\frac{3\pi}{2n}+\cos\frac{3\pi}{2n}}+\frac{1}{5}\frac{1}{\cosh\frac{5\pi}{2n}+\cos\frac{5\pi}{2n}}-\ldots,$$

we have

$$-16\,i\,S+2\pi i=\int_0^{2\pi}\frac{1}{\cosh\frac{z}{n}+\cos\frac{z}{n}}\left(\frac{1}{\cosh z}+\frac{1}{\cos z}\right)i\,d\theta,$$

where $z=Re^{i\theta}=\alpha+i\beta$.

Now the functions

$$2\cosh z=e^{\alpha+i\beta}+e^{-\alpha-i\beta}$$

$$2\cos z=e^{i\alpha-\beta}+e^{-i\alpha+\beta}$$

$$2\left(\cosh\frac{z}{n}+\cos\frac{z}{n}\right)=e^{\frac{\alpha+i\beta}{n}}+e^{-\frac{\alpha+i\beta}{n}}+e^{\frac{\beta-i\alpha}{n}}+e^{-\frac{\beta-i\alpha}{n}}$$

all have an infinitely great modulus in general, the exceptional cases are

(1) $\beta=0$, when $\cos z=\pm 1$ by the above choice of $R=k\pi$ or $(k+1)\pi$ and $\cosh \frac{z}{n}+\cos \frac{z}{n}$ is infinite,

(2) $\alpha=0$, when $\cosh z=\pm 1$ and again $\cosh \frac{z}{n}+\cos \frac{z}{n}$ is infinite,

(3) $\alpha=\beta$, when $\cosh \frac{z}{n}+\cos \frac{z}{n}$ is not zero by the above choice of R and $\cosh z$ and $\cos z$ are both infinite.

Thus every element of the integral on the right is zero and therefore $S=\frac{\pi}{8}$.

Question 652.

(S. P. Singaravelu Mudaliar) :—From a point (eccentric angle ϕ) of an ellipse of semi-axes a, b, three normals are drawn to the ellipse ; show that the square on the radius of the circle passing through the feet of the normals is

$$\left(a+\frac{b^2}{2a}\right)^2\cos^2\phi+\left(b+\frac{a^2}{2b}\right)^2\sin^2\phi.$$

Additional solution by K. B. Madhava.

Let the equation of the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ... (1)

Then the equation of the rectangular hyperbola which passes through the feet of the normals from the point (ϕ) is

$$(a^2-b^2)\,xy+b^3\sin\phi\,x-a^3\cos\phi\,y=0\,; \quad \ldots \quad (2)$$

so that eliminating x, we get as the equation giving the ordinates of the four feet of the normals from (ϕ)—one of which of course is ϕ itself,

$$y^4(a^2-b^2)^2+2b^3\sin\phi\,(a^2-b^2)y^3+y^2[(a^4b^2\cos^2\phi+b^6\sin^2\phi$$
$$-b^2(a^2-b^2)^2]-2b^5\sin\phi\,(a^2-b^2)y-b^8\sin^2\phi=0 \quad \ldots \quad (3)$$

Therefore rejecting the root 'ϕ' of this equation, the equation giving the ordinates of the three other points is

$$y^3(a^2-b^2)^2+b\sin\phi\,(a^2-b^2)(a^2+b^2)y^2+b^4(2a^2-b^2)y+$$
$$b^7\sin\phi=0. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Now let us suppose that

$$x^2+y^2+2gx+2fy+c=0 \quad \ldots \quad \ldots \quad (5)$$

is the circle which passes through these three points. Eliminating x, between (5) and (1), we get as the equation giving the ordinates of these three points and that of a fourth point of intersection

$$y^4(a^2-b^2)^2-4fb^2(a^2-b^2)y^3+y^2(4f^2b^4+2b^2(c+a^2)(b^2-a^2)+4b^2a^2g^2)$$
$$+4fb^4(c+a^2)y+b^4(c+a^2)^2-4g^2a^2b^4=0. \quad \ldots \quad \ldots \quad (6)$$

But it is known that this fourth point is diametrically opposite to the foot of the fourth normal which in this case is 'ϕ'.

Multiplying (4) therefore by the factors $(y+b\sin\phi,)$ we get a quartic which can be identified with (6). Making the necessary reductions we have

$$f=-\frac{a^2}{2b}\sin\phi=-\frac{b^2}{2a}\cos\phi\, c=-(a^2+b^2). \qquad (7)$$

Hence the equation of the circle (5) is

$$x^2+y^2-\frac{b^2}{a}x\cos\phi-\frac{a^2}{b}y\sin\phi=a^2+b^2.$$

The square on its radius can be put in the form

$$\left(a+\frac{b^2}{2a}\right)^2\cos^2\phi+\left(b+\frac{a^2}{2b}\right)^2\sin^2\phi.$$

Question 675.

(K. Appukuttan Erady) :—Show that

$$(1)\ \int_0^{\pi}\frac{e^{b\cos\theta}\cos(b\sin\theta)}{1-2a\cos\theta+a^2}\,d\theta=\frac{\pi\, e^{ab}}{1-a^2};$$

$$(2)\ \int_0^{\pi}\frac{\log(1-2b\cos\theta+b^2)}{1-2a\cos\theta+a^2}\,d\theta=\frac{2\pi}{1-a^2}\log(1-ab).$$

Solution by K. B. Madhava.

The question is not rigorously worded; it is obvious that we should have $o<a<1$ in either case, and $o<(b)a<1$ in the second, in addition. There are various ways of establishing these well known results. We will employ the method of contour integration so as to bring out prominently the necessity for the limits of a and b.

Put $z=\cos\theta$ and corresponding to the interval 0 to 2π for θ, we shall have as the region of the variable z, a circle (c) of radius (a) about the origin and then the integral

$$(1)=\tfrac{1}{2}\int_0\frac{dz\ e^{bz}}{i\,(1-az)(z-a)}$$

which has a simple pole at $z=a$, if and only if $o<a<1$. (A).

For the residue there

$$\lim_{z\to a}\frac{(z-a)e^{bz}}{i(z-a)(1-az)}=\frac{e^{ab}}{i(1-a^2)}$$

and hence by Cauchy's theorem, the given integral

$$(1)\ =\tfrac{1}{2}\cdot 2\pi i\cdot\frac{e^{ab}}{i(1-a^2)}=\frac{\pi e^{ab}}{1-a^2}$$

by equating real parts.

Again, for the second part, the integral

$$=\int_0^{2\pi} \frac{\frac{1}{2}\log(1-2b\cos\theta+b^2.)}{1-2a\cos\theta+a^2.}d\theta$$

which by the same transformation as before

$$=\text{real part of}\int_0 \frac{\log(1-bz).}{i(1-az)(z-a)}dz.$$

which has got as its only pole $z=a$, a simple pole; if (A) is satisfied and in addition $o<ba<1$. (B).

Evaluating the residue as before and applying Cauchy's theorem, the integral in question

$$=\frac{2\pi}{1-a^2}\log(1-ab).$$

We thus see the necessity for conditions (A) & (B).

Question 697.

(S. Krishnasawmi Aiyangar) :—The base BC of a triangle is given in magnitude and position. If the intercept on the tangent to the incircle parallel to the base and terminated by the sides is of constant length, find the locus of the vertex.

Solution by Martyn M. Thomas, R. D. Karve and others.

Let k be the constant intercept; then

$$k=r\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$$

$$=\frac{r\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$=2R\sin A\cdot\tan\frac{B}{2}\tan\frac{C}{2}$$

$$=a.\frac{s-a}{s}.$$

Since k and a are constant, it follows that s is constant, and therefore $b+c$ is constant.

Hence the locus of A is an ellipse with foci at B and C.

Questions 702 and 715.

(MARTYN. M. THOMAS) :—Prove that

$$\text{(i)} \int_0^{\frac{\pi}{2}} \sin x\,(\log \sin x) \cos x\,(\log \cos x)\, dx = \frac{12-\pi^2}{48};$$

$$\text{(ii)} \int_0^{\infty} \frac{\sin x \log(\cos x)}{x} dx = \pi \log \frac{1}{\sqrt{2}}.$$

[*N.B.*—Q. 715 is only a repetition of Q. 702 (i).]

Solution by J. M. Bose, M.A., B. Sc., D. Krishnamurti, K. R. Rama Iyer and others.

$$\text{(i)} \int_0^{\frac{\pi}{2}} \sin x \log(\sin x) \cos x \log(\cos x)\, dx.$$

The required integral becomes

$$\frac{1}{8}\int_0^1 \log z \log(1-z)\, dz, \text{ on putting } \sin x = z^{\frac{1}{2}}.$$

Now

$$\frac{1}{8}\int_0^1 \log z \log(1-z)\, dz = -\frac{1}{8}\int_0^1 \sum_1^{\infty} \frac{z^n \log z}{n} dz$$

$$= -\frac{1}{8}\left[\sum_1^{\infty} \frac{z^{n+1}}{n(n+1)}\left(\log z - \frac{1}{n+1}\right)\right]_0^1;$$

and it is easy to prove

$$\underset{z=0}{\mathrm{L}}\; z^r \log z = 0.$$

Hence the integral becomes

$$\frac{1}{8}\sum \frac{1}{n(n+1)^2} + \frac{1}{8}\sum_1^{\infty}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)^2}\right)$$

$$= \frac{1}{8}\left(1 - \frac{\pi^2}{6} + 1\right) = \frac{12-\pi^2}{48},$$

Hence $\qquad \sum \frac{1}{n(n+1)} = \frac{1}{1.2} + \frac{1}{2.3} + \ldots\infty = 1.$

and $\qquad \sum \frac{1}{(n+1)^2} + 1 = \frac{1}{1^2} + \frac{1}{2^2} + \ldots\infty = \frac{\pi^2}{6}.$

(ii) $\int_0^\infty \frac{\sin x \log \cos x \, dx}{x}$.

We have

$$\log \cos x = -\log 2 + \sum_1^\infty \frac{(-1)^{n+1}}{n} \cos 2nx.$$

Hence

$$\int_0^\infty \frac{\sin x \log \cos x}{x} dx = -\log 2 \int_0^\infty \frac{\sin x}{x} x$$

$$+ \sum_1^\infty \frac{(-1)^{n+1}}{n} \int_0^\infty \frac{\sin x \cos 2nx}{x} dx$$

$$= -\log 2 \int_0^\infty \frac{\sin x}{x} dx +$$

$$\tfrac{1}{2} \sum^\infty \frac{(-1)^{n+1}}{n} \left[\int_0^\infty \frac{\sin (2n+1)x}{n} dx + \int_0^8 \frac{\sin (1-2n)x}{x} dx \right]$$

$$= -\log 2 \cdot \frac{\pi}{2} + \frac{1}{2} \sum_1^\infty \frac{(-1)^{n+1}}{n} \left[\frac{\pi}{2} + \frac{\pi}{2} \right]$$

$$= -\frac{\pi}{2} \log 2 + \frac{\pi}{2} \sum \frac{(-1)^{n+1}}{n}$$

$$= -\frac{\pi}{2} \log 2 \, ; \text{ since } \sum_1^\infty (-1)^{n+1} \cdot \frac{1}{n} = 0.$$

Question 704.

(S. Malhari Rao, B. A.) :—If the sum of a number of three digits and the number formed by reversing the digits be divisible by 37, shew that the sum of all such pairs of number is 480×37.

Solution by K. J. Sanjana and N. S. Rajvade.

If the digits of any number be x, y, z, the sum of this and the number formed by reversing the digits is $101x + 20y + 101z$.

Now $101x + 20y + 101z = 111 \, ((x+z) - 10 \, (x+z-2y)$, which will be a multiple of 37 when $x+z=2y$, in which case the sum of this pair of numbers is $222y$.

If $x \neq z$, it is seen that when $x=1$, $z=3$ or 5 or 7 or 9, and y is 2, 3, 4 or 5; when $x=2$, $y=3$, 4 or 5; when $x=3$, $y=4$, 5 or 6, omitting numbers already accounted for. Similarly, when x is 4, 5, 6, 7, y has the values

5 or 6, 6 or 7, 7, 8;

and there are no further numbers. The sum of all the possible values of y being 80, the sum of all the numbers found is $222\times 80=37\times 480$.

When $x=z$, x, y, z are all equal; in this case we get the numbers (unaltered by reversion of digits) to be 111, 222,......999, whose sum is $3\times 37\ (1+2...+9)=37\times 135$. The proposer has evidently excluded these from consideration.

Additional solution by K. B. Madhava.

Question 706.

(N. Sankara Aiyar, M.A.) :—If n is any number, show that

$$\frac{1}{(n+1)(n+2)}+\frac{1}{2}\frac{1}{(n+2)(n+3)}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{(n+3)(n+4)}+\ldots$$
$$=2^{2n+2}\frac{\Gamma(n+1)\Gamma(n+2)}{\Gamma(2n+4)}.$$

Solution by K. Appukuttan Erady, K. B. Madhava and K. J. Sanjana, M.A.

The left hand side is the difference of the two series

$$\frac{1}{n+1}+\frac{1}{2}\frac{1}{n+2}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{n+3}+\ldots$$

and

$$\frac{1}{n+2}+\frac{1}{2}\frac{1}{n+3}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{n+4}+\ldots,$$

each of which is absolutely convergent.

Hence the given expression

$$=\int_0^1\left(x^{n}+\frac{1}{2}x^{n+1}+\frac{1\cdot 3}{2\cdot 4}x^{n+2}\ldots\right)dx$$
$$-\int_0^1\left(x^{n+1}+\frac{1}{2}x^{n+2}+\frac{1\cdot 3}{2\cdot 4}x^{n+3}\ldots\right)dx$$

$$=\int_0^1 x^{n}(1-x)^{-\frac{1}{2}}dx-\int_0^1 x^{n+1}(1-x)^{-\frac{1}{2}}dx$$

$$= \int_0^1 x^n (1-x)^{\frac{1}{2}} dx = B\left(n+1, \tfrac{3}{2}\right)$$

$$= \frac{\Gamma(n+1)\Gamma(\frac{3}{2})}{\Gamma(n+\frac{5}{2})} = \frac{1}{2}\frac{\Gamma(n+1\sqrt{\pi}}{\Gamma(n+5/2)}.$$

But

$$\Gamma(n+2)\,\Gamma(n+2+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2n+3}}\Gamma(2n+4). \quad \text{(Williamson, p. 164).}$$

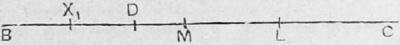
Hence the result.

Question 709.

(R. Srinivasan, M. A.) :—The common tangent to the nine points circle of a triangle ABC and the ex-circle opposite to A meets BC in D ; E and F are corresponding points for CA and AB. Show that AD, BE, CF are concurrent.

Solution (1) *by K. J. Sanjana* ; (2) *by K. Appukuttan Erady.*

(1) Let the ex-circle touch BC in X_1 and the nine-points circle in T ; take M the mid-point of BC and L the first of the perpendicular on it from A.



Then $DX_1 = DT$ from the ex-circle, and $DT^2 = DM.DL$ from the nine-points circle. Hence $DX_1^2 = DM \cdot DL$;

$$\therefore \quad \frac{DX_1}{DM} = \frac{DL}{DX_1} = \frac{X_1L}{X_1M}, \text{ so that } \frac{BD-(s-c)}{\frac{1}{2}a - BD} = \frac{c\cos B-(s-c)}{\frac{1}{2}a-(s-c)}.$$

The right side reduces to $2s/a$, on multiplying by $2a$ above and below ; hence $BD\,(a+2s) = sa + sa - ac = a^2 + ab$,

so that
$$BD = \frac{a(a+b)}{2a+b+c}.$$

$$\therefore \quad CD = \frac{a(a+c)}{2a+b+c}, \text{ and } \frac{BD}{CD} = \frac{a+b}{a+c}.$$

Similar results hold for

$$\frac{CE}{AE} \text{ and } \frac{AF}{BF};$$

and the product of the three being unity, AD, BE and CF are concurrent.

If the common tangent of the nine-points circle and the in-circle cuts BC in P, we can show similarly that

$$BP : CP = b-a : c-a.$$

Of course P and the two similar points for CA, BA, are collinear.

(2) Let the equation to the ex-circle opposite to the angle A be

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta + (l\alpha + m\beta + n\gamma)(a\alpha + b\beta + c\gamma) = 0,$$

where this meets $\alpha=0$ we must have

$$\frac{\beta}{c(s-b)}=\frac{\gamma}{b(s-c)};$$

Hence

$$\frac{m}{b(s-c)}+\frac{n}{c(s-b)}+\frac{1}{bc}=0 \qquad \dots \qquad \dots \quad (1)$$

Again where the circle meets $\beta=0$ we must have

$$\frac{\alpha}{c(s-b)}=\frac{\gamma}{-as};$$

hence

$$\frac{l}{as}-\frac{m}{b(s-c)}+\frac{1}{ab}=0; \qquad \dots \qquad \dots \quad (2)$$

and the corresponding equation obtained by finding where the circle cuts $\gamma=0$, *viz.*,

$$\frac{l}{as}-\frac{m}{b(b-c)}+\frac{1}{ab}=0. \qquad \dots \qquad \dots \quad (3)$$

Solving these equations we get

$$l=-\frac{s^2}{bc},\ m=-\frac{(s-c)^2}{ac},\ n=-\frac{(s-b)^2}{ac}.$$

Now the equation to the nine-points circle can be put in the form

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+(p\alpha+q\beta+r\gamma)(a\alpha+b\beta+c\gamma)=0$$

where $p=\dfrac{a^2-b^2-c^2}{4bc}$, and corresponding values for q and r.

Now the equation to the radical axis of these two circles is

$$(p-l)\alpha+(q-m)\beta+(r-n)\gamma=0;$$

which, when the above values for l,m,n,p,q,r are substituted, reduces to

$$\frac{a\alpha}{b-c}+\frac{b\beta}{a+c}-\frac{c\gamma}{a+b}=0,$$

or in areal co-ordinates

$$\frac{x}{b-c}+\frac{y}{a+c}-\frac{z}{a+b}=0.$$

Hence the common tangent of the nine-point circle and the ex-circle opposite to A divides BC in the ratio of $(a+b):(a+c)$.

Similarly, the other common tangents corresponding to the other ex-circles divide the sides C A, A B in the ratios $b+c:b+a$ and $a+c:a+b$, respectively. The product of three ratios being unity the straight lines joining these points of division to the opposite vertices are concurrent.

QUESTIONS FOR SOLUTION.

ERRATUM.

549. *Corrected* —(K. J. SANJANA, M.A.) :—If

$$S_n^r = 1^r - 2^r + 3^r - \ldots\ldots (-)^{n+1} n^r,$$

prove that

$$S_n^r + \binom{r+1}{1} S_{n-1}^r + \binom{r+1}{2} S_{n-2}^r + \binom{r+1}{3} S_{n-3}^r + \ldots = 0,$$

when r is even and n is any integer equal to or greater than r. [When $n = r-1$, the result is unity.] Shew also how the value of the series may be found when r is odd.

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757. (S. MALHARI RAO, B. A.) :—Give a solution in positive integers of the equations (1) $x^4 - 6y^2 = 1$, (2) $x^{13} - 17y^2 = x^6$.

758. (S. MALHARI RAO, B. A.) :—If the integers x, y, z represent the sides of a right-angled triangle and x, z are primes greater than 5, shew that y is a multiple of 60 ; and that $x + z = 1800$, when $y = 1740$.

759. (K. APPUKUTTAN ERADY, M. A.) :—Three circles of radii a, b, c touch one another externally. If the radii of the circles touching them be r, r' $(r < r')$, show that $r, -r'$ are the roots of

$$\rho^2 . \left\{ 2\Sigma\left(\frac{1}{bc}\right) - \Sigma\left(\frac{1}{a^2}\right) \right\} + 2\rho . \Sigma\left(\frac{1}{a}\right) - 1 = 0.$$

760. (K. APPUKUTTAN ERADY, M. A.) :—If

$$f(n) \equiv \frac{1}{n} + \frac{1}{n(n+1)} + \frac{1}{n(n+1)(n+2)} + \ldots\ldots$$

show that

$$f(1) + \frac{f(2)}{1!} + \frac{f(3)}{2!} + \ldots\ldots = e.$$

761. (MARTYN, M. THOMAS, M.A.) :—Apply the method of the *Instantaneous Centre* to solve the following :—

At every point of a plane curve a line is drawn making a given angle α with the normal, and let the envelope of these lines be termed the α—evolute. Prove that the β—evolute of the α—evolute of any curve is the α—evolute of the β—evolute of the curve.

[MADRAS, M. A. 1911.]

762. (Professor K. J. Sanjana) :—Two equal uniform rods AB, BC, rigidly jointed at B at an angle 2α, float in a liquid in an unsymmetrical position, with the angle B outside the fluid and A and C immersed : prove that the ratio of the specific gravity of the rods to that of the liquid must lie between

$$\frac{1+\sin^2\alpha}{2} \text{ and } \frac{2\sin^2\alpha}{1+\sin^2\alpha}.$$

If the angle B is immersed and A and C are outside, the limits are

$$\frac{\cos^2\alpha}{1+\sin^2\alpha} \text{ and } \frac{\cos^2\alpha}{2}.$$

763. (S. R. Ranganathan, B.A., Hon.) :—A comet is moving along an arc of a parabola, which has the Sun at its focus. Show that the amount of heat it receives from the Sun in any interval is proportional to the angle through which its direction of motion changes during that interval.

764. (S. R. Ranganathan, B.A., Hon.) :—Find the form of a solid of revolution, into which a given quantity of matter of uniform density should be put so that its moment of inertia about a diameter of the equatorial plane may be a minimum.

765. (Martyn M. Thomas, M.A.) :—Find the simplest Cartesian curve in which, if the abscissae increase by unity, the radii of curvature increase in the ratio of the corresponding slope of the curve.

766. (S. Krishnaswami Iyengar) :—Find the value of

$$\frac{1}{1}+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\ldots\ldots+\frac{1}{n^n}+\ldots\ldots$$

767. (S. Krishnaswami Iyengar) :—Show that

$$1-\frac{1}{2^2\cdot 5}+\frac{1}{2^4\cdot 9}-\frac{1}{2^6\cdot 13}\cdots(-)^m\frac{1}{2^{2n}\cdot(4n+1)}\cdots$$
$$=\tfrac{1}{4}\log 5+\tfrac{1}{2}\tan^{-1} 2.$$

768. (S. Ramanujan) :—If $\psi(x)=\dfrac{x+2}{x^2+x+1}$,

show that

$$\text{(i)}\quad \frac{1}{3}\psi(x^{\frac{1}{3}})+\frac{1}{9}\psi(x^{\frac{1}{9}})+\frac{1}{29}\psi(x^{\frac{1}{27}})+\ldots=\frac{1}{\log x}+\frac{1}{1-x}$$

for all positive values of x; and

$$\text{(ii)}\quad \frac{1}{3}\psi(x^{\frac{1}{3}})+\frac{1}{9}\psi(x^{\frac{1}{9}})+\frac{1}{27}\psi(x^{\frac{1}{27}})+\ldots=-\frac{1}{1-x}$$

for all negative values of x.

769. (S. RAMANUJAN) :—Show that

$$\log 2\left(\frac{1}{2\log 2\log 4}+\frac{1}{3\log 3\log 6}+\frac{1}{4\log 4\log 8}+\ldots\right)$$

$$+\frac{1}{2\log 2}-\frac{1}{3\log 3}-\frac{1}{4\log 4}-\frac{1}{5\log 5}+\ldots\ldots\ldots=\frac{1}{\log 2}.$$

770. (S. RAMANUJAN) :—If δ_n denote the number of divisors of n (*e.g.*, $\delta_1=1$, $\delta_2=2$, $\delta_3=2$, $\delta_4=3$,...), show that

$$\text{(i)}\quad \frac{\delta_1}{1}-\frac{\delta_3}{3}+\frac{\delta_5}{5}-\frac{\delta_7}{7}+\frac{\delta_9}{9}\ldots\ldots\ldots$$

is a convergent series ; and

$$\text{(ii)}\quad \frac{\delta_1}{1}-\frac{\delta_2}{2}+\frac{\delta_3}{3}-\frac{\delta_4}{4}+\frac{\delta_5}{5}\ldots\ldots\ldots$$

is a divergent series in the *strict* sense (*i.e.*, not oscillating).

771. (K. B. MADHAVA, B. A., Hon.) :—In his solution to Q. 629, Mr. Bhimasena Rao, uses Bromwich P. 447, Ex. 4, (after the method of Dirichlet's integrals), to show that if

$$\psi(t)=\sum_{n=-\infty}^{\infty} e^{-n^2\pi t}$$

we shall have

$$\psi(t)=t^{-\frac{1}{2}}\psi\left(\frac{1}{t}\right).$$

This result is due to Jacobi. See *Ges-Werke.* II, p. 188.

Verify this by integrating $\int \frac{e^{-z^2\pi t}}{e^{2\pi i z}-1}\,dz$ round a rectangle whose vertices are $\pm(\mathrm{R}+\frac{1}{2})\pm i$ and making $\mathrm{R}\rightarrow\infty$ in the usual manner.

Show also by the same method, that if $t>0$,

$$\sum_{-\infty}^{\infty} e^{-n^2\pi t-2n\pi at}=t^{-\frac{1}{2}}e^{\pi a^2 t}\left\{1+2\sum_{1}^{\infty} e^{-\frac{n^2\pi}{t}}\cos 2n\pi a\right\},$$

also due to Jacobi.